Introduction to New Banach Function Spaces Theory

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Analysis in Tatra- Seminar for students

Male Ciche

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[CF] **C. Capone -M.R.Formica**, *A Decomposition of the Dual Space of Some Banach Function Spaces*, Journal of Function Spaces and Applications, 2012.

Outline

- L^p spaces
- Introduction to BFS theory
- Examples
- A decomposition formula in rearrangement invariant BFS

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- $\Omega \subset \mathbb{R}^n$ measurable
- $1 \leq p < \infty$

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}\,,\ f\,\text{measurable}\,,\ \int_\Omega |f|^p dx < +\infty \right\}$$

- Let us set

$$ess\ \sup_\Omega |f| = \inf\{C > 0\ :\ |f(x)| \leq C\ a.e.\ in\,\Omega\} \Leftrightarrow \text{essential sup of } f$$

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R}\,,\ f\,\text{measurable}\,,\ ess\ \sup_\Omega |f| < +\infty \right\}$$

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Some properties

- Minkowsky inequality : Let $1 < p < \infty$

$$||f+g||_{L^p(\Omega)} \leq ||f||_{L^p(\Omega)} + ||g||_{L^p(\Omega)}$$

- Conjugate exponents: Let $1 \leq p \leq \infty$, $1 \leq p' \leq \infty$,

$$\frac{1}{p} + \frac{1}{p'} = 1$$

- Hölder inequality: Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$

$$\int_\Omega |f \cdot g| dx \leq ||f||_{L^p(\Omega)} \cdot ||g||_{L^{p'}(\Omega)}$$

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- Dominated convergence theorem

1. If $f_n \to f$ a.e., $\exists\, g \in L^1(\Omega)$ s.t., and $|f_n| \leq g$ for all n, a.e. in Ω.

   Then f is integrable and

$$\int_\Omega f dx = \int_\Omega \lim_{n\to\infty} f_n \, dx = \lim_{n\to\infty} \int_\Omega f_n \, dx$$

- Monotone convergence theorem

1. If $f_n : \Omega \to [0, +\infty)$ is an increasing sequence of measurable functions which converges pointwise a.e.to f, then

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- Fatou's lemma

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$$\int_\Omega \liminf_{n\to\infty} f_n dx \le \liminf_{n\to\infty} \int_\Omega f_n dx$$

- Fisher-Riesz Theorem

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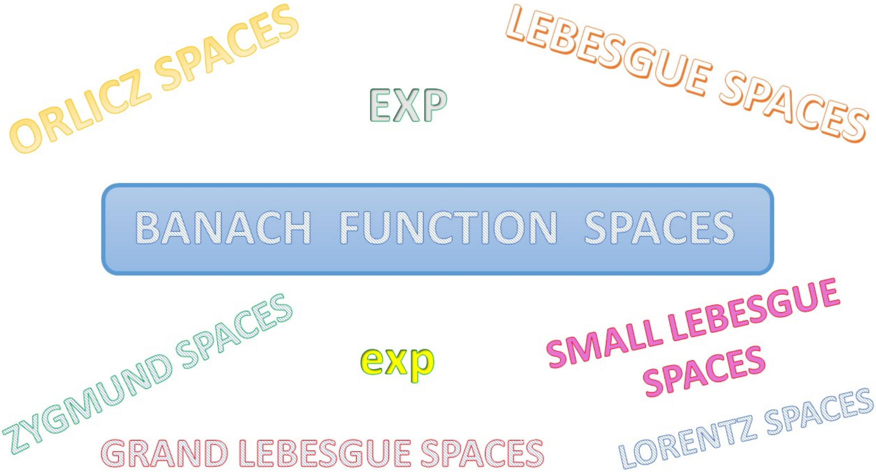
Introduction to BFS theory



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- (Ω, μ) a measure space

- M_0^+ the set of all measurable functions with values in $[0, \infty]$, finite a.e. in Ω

- χ_E the characteristic function of a measurable subset $E \subset \Omega$

- (Ω, μ) a measure space, $A \subset \Omega$, $\mu(A) > 0$, A *atom* if it does not contain any measurable subset with positive measure.

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BFS Axioms

A function $\rho : M_0^+ \to [0, \infty]$ is a Banach function norm (simply a function norm) if, for all f, g, $f_n\,(n, 1, 2, ...)$ in M_0^+ the following properties hold

- $\rho(f) = 0 \quad \Leftrightarrow \quad f = 0 \quad \mu-$a.e.
- $\rho(\alpha f) = \alpha \rho(f)$
- $\rho(f + g) \leq \rho(f) + \rho(g)$
- $0 \leq g \leq f \;\; \mu - a.e. \Rightarrow \rho(g) \leq \rho(f)$ (Lattice property)
- $0 \leq f_n \uparrow f \;\; \mu - a.e. \Rightarrow \rho(f_n) \uparrow \rho(f)$ (Fatou property)
- $\mu(E) < \infty \quad \Rightarrow \quad \rho(\chi_E) < \infty$
- $\mu(E) < \infty \quad \Rightarrow \quad \int_E f\, d\mu < C_E\, \rho(f)$

  with C_E dependent on ρ, but independent on f.

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Fatou property

- Validity of a monotone convergence Theorem
- Fatou Lemma has an analogous in every BFS
- Riesz-Fisher property * rf   torna a rf

$$X = X(\rho) \text{ BFS}, \; f_n \in X, \; (n = 1, 2, ..), \; \sum_{n=1}^{\infty} ||f_n||_X < \infty$$

Then $\displaystyle\sum_{n=1}^{\infty} f_n = f \in X$ and

$$||f||_X \leq \sum_{n=1}^{\infty} ||f_n||_X$$ Riesz-Fisher property

1. Completeness of any BFS

ESEMPIO

$$\rho(f) = \begin{cases} \left(\int_\Omega f^p d\mu \right)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \underset{\Omega}{esssup}\, f & (p = \infty) \end{cases}$$

LEBESGUE FUNCTIONALS

Banach Function Space

Let ρ a function norm, the Banach Function Space, $X = X(\rho)$ is the set of all measurable functions f such $\rho(f) < \infty$, and, for all $f \in X$, let us set

$$||f||_X = \rho(f)$$

$$\rho(f) \text{ Lebesgue Functionals} \quad \Rightarrow \quad X(\rho) = L^p(\Omega)$$

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Associate norm

Let ρ a function norm the associate norm is the function defined as

$$\rho'(f) = \sup\left\{\int_{\Omega} |fg|\, d\mu : f \in M_0^+, \rho(f) \leq 1\right\}$$

ρ function norm $\Rightarrow$ ρ' function norm

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Associate space

- ρ function norm
- $X(\rho)$ BFS
- $\rho^{'}$ associate norm

The associate space to $X(\rho)$ is the BFS $X^{'} = X^{'}(\rho^{'})$ associate to $\rho^{'}$

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Theorem

Let $X = X(\rho)$ be a BFS and $X^{'} = X^{'}(\rho^{'})$ the associate space. A measurable function $g \in X^{'}$ if and only if fg is integrable for all $f \in X$ and

$$\int_\Omega fg\, d\mu \leq ||f||_X\, ||g||_{X'} \qquad (H)$$

$$f \in X\,, g \in X^{'}, \, ||f||_X > 0 \Rightarrow \left|\left|\frac{f}{||f||_X}\right|\right|_X = 1 \Rightarrow \int_\Omega \left|\frac{f}{||f||_X} g\right| dx \leq ||g||_{X'}$$

Let fg integrable, if $\rho^{'}(|g|) = \infty \Rightarrow \exists\, f_n \,:\, ||f_n||_X \leq 1\,, \int_\Omega f_n g > n^3$

Since $\sum_{n=1}^{\infty} ||n^{-2} f_n|||_X < \infty$, then, by Riesz-Fisher property, * rf

$$f = \sum_{n=1}^{\infty} n^{-2} f_n \in X \Rightarrow \int_\Omega |fg| dx > n^{-2} \int_\Omega |f_n g| dx > n \quad , \quad \forall n$$

The contradiction proves that $g \in X^{'}$ and this completes the proof.

Theorem (Lorentz-Luxemburg)

Let $X = X(\rho)$ be a BFS, then $X \equiv X''$ and

$$||f||_X = ||f||_{X''}$$

Let $\phi : [0, +\infty) \to [0, +\infty)$

- increasing
- right continuous
- $\phi(0) = 0 \quad \lim_{t\to\infty} \phi(t) = \infty$

$$\Phi(t) = \int_0^t \phi(s)ds \qquad \mathrm{N-function}$$

- continuous
- convex
- increasing
- $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0 \ , \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$

Examples of BFS

Let $\phi : [0, +\infty) \to [0, +\infty)$

- increasing
- right continuous
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$$L^{\Phi}(\Omega) = \left\{ f \in L^1(\Omega) : \exists \lambda > 0 : \fint_{\Omega} \Phi\left(\frac{|f|}{\lambda}\right) < \infty \right\}$$ ORLICZ SPACE

$$||f||_{\Phi} = \inf \left\{ \lambda > 0 : \fint_{\Omega} \Phi\left(\frac{|f|}{\lambda}\right) \leq 1 \right\}$$ LUXEMBURG NORM

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Orlicz Spaces

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  LEBESGUE SPACES

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SPACE OF EXPONENTIALLY
INTEGRABLE FUNCTIONS

$$\left(L^{\Phi}(\Omega)\right)' = L^{\tilde{\Phi}}(\Omega)$$

ASSOCIATE ORLICZ SPACE

$$\tilde{\Phi} = max\{st - \Phi(s) \, : \, s \geq 0\}$$

COMPLEMENTARY FUNCTION OF Φ

GRAND LEBESGUE SPACES

([Iwaniec-Sbordone])

Let $1 < p < \infty$

$$L^{p)}(\Omega) = \left\{ f \in L^1(\Omega) : ||f||_{p)} < \infty \right\}$$

where

$$||f||_{p)} = \sup_{0<\epsilon<p-1} \left(\epsilon \fint_{\Omega} |f|^{p-\epsilon} \right)^{\frac{1}{p-\epsilon}}$$

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GENERALIZED GRAND LEBESGUE SPACES

([C.-Fiorenza])

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SMALL LEBESGUE SPACE

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Let $1 < p < \infty$

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and the auxiliary space $L^{(p'}(\Omega)$ defined as

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$$\left(L^{(p',\theta}(\Omega) \, , \|f\|_{(p',\theta} \right) \overset{?}{\equiv} \left(L^{p)',\theta}(\Omega) \, , \|f\|_{p)',\theta} \right)$$

[C. −Fiorenza]

Small Lebesgue space

$$L^{(p',\theta}(\Omega) = \left\{ f \in L^1(\Omega) : ||f||_{(p',\theta} < \infty \right\}$$

$$||f||_{(p',\theta} = \sup \left\{ \int_\Omega |fg| d\mu \ : ||g||_{p)',\theta} \leq 1 \right\}$$

$$\left(L^{(p',\theta}(\Omega) \, , \, ||f||_{(p',\theta} \right)$$

SMALL LEBESGUE SPACE

ASSOCIATE SPACE OF $L^{p),\theta}(\Omega)$

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Gossez, Marcellini, Mascolo, Mingione, Wang, Zhao.....])

DUALE E ASSOCIATO DI UN BFS

Dual and associate space of a BFS

Theorem

The associate space X' of a BFS X is canonically isometrically isomorph to a closed subspace norm-fundamental of the Banach dual space X^*. * d1

Definition

Let X a BFS, $f \in X$ has absolute continuous norm if

$$||f\chi_{E_n}|| \to 0\,, \ \ \forall \{E_n\}_{n=1}^{\infty} \ : \ E_n \to \emptyset \text{ a.e.}$$

$$X_a = \{f \in X : f \text{ has absolute continuous norm}\}$$

X has absolute continuous norm if $X = X_a$.
(where $E_n \to \emptyset$ if $\chi_{E_n} \to 0 \ \ \mu - \text{a.e.}$)

Definition

Let X be a BFS let us denote by X_b the closure of the set of simple functions.

Dual and associate space of a BFS

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Definition

Let X be a BFS let us denote by X_b the closure of the set of simple functions.

Properties

- $X_a \subseteq X_b \subseteq X$
- X_b is isometrically isomorph to a subspace of $(X')^*$
- $X_a \equiv X_b \;\Leftrightarrow\; \chi_E$ a.c.n. $\forall\, E\,,\; \mu(E) < \infty$

  * d2
- X_a and X_b, are order ideal
- If X_a contains the simple functions then $(X_a)^* = X'$

  * p1 torna a p1
- 1. $X^* \simeq X' \;\Leftrightarrow\; X = X_a$

  * p2 torna a p2
- 2. X reflexive BFS $\;\Leftrightarrow\; X = X_a$ and $X' = X'_a$

Example:

$$(L^{(p',\theta}(\Omega))' = (L^{p)',\theta}(\Omega))' = (L^{p),\theta}(\Omega))'' = L^{p),\theta}(\Omega)$$

[C. −Fiorenza] — Lorentz-Luxemburg Theorem — X BFS, $X = X''$

$$(L^{(p',\theta}(\Omega))' = L^{p),\theta}(\Omega)$$

On the other hand

$$L^{(p',\theta}(\Omega) = (L^{(p',\theta}(\Omega))_a$$

[C. −Fiorenza]

p1

$$(L^{(p',\theta}(\Omega))^* = (L^{(p',\theta}(\Omega))' = L^{p),\theta}(\Omega)$$

p1 — Previous identity

$$(L^{(p',\theta}(\Omega))^* = L^{p),\theta}(\Omega)$$

.

Applications

$$\mathrm{LlogL}(\Omega) = (\mathrm{LlogL}(\Omega))_a$$

$\Downarrow$ p1

$$(\mathrm{LlogL}(\Omega))^* = (\mathrm{LlogL}(\Omega))'$$

$\Downarrow$

$$(\mathrm{LlogL}(\Omega))^* \overset{\text{p1}}{=} (\mathrm{LlogL}(\Omega))' \underset{\text{[BS]}}{=} ((\mathrm{EXP}(\Omega))')' = (\mathrm{EXP}(\Omega))'' \overset{\text{Teo. Lorentz-Luxemburg}}{\underset{X \text{ BFS},\ X = X''}{=}} \mathrm{EXP}$$

Let $f \in M_0^+$

$$\mu_f(\lambda) = \mu\ \{x \in \Omega : |f(x)| > \lambda\}$$

$\Updownarrow$ Def

distribution function

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$$

$\Updownarrow$ Def

decreasing rearrangement

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds \quad (t > 0)$$

$\Rightarrow$

f^{**} *non increasing*

$$f^*(t) \leq f^{**}(t), \quad t > 0$$

Definition

The functions f , $g \in M_0^+$ are equimisurable if $\mu_f(\lambda) = \mu_g(\lambda)$.

Definition

A function norm ρ is said rearrangement invariant if $\rho(f) = \rho(g)$, for all equimisurable functions f and g.

Definition

The BFS $X(\rho)$ is a rearrangement invariant space (r.i.s.) if ρ is a rearrangement invariant norm.

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Examples

Lebesgue Spaces

Orlicz Spaces

Grand Lebesgue Spaces

Small Lebesgue Spaces

Last tools: Direct Sum, Orthogonal Space, Fundamental Function

Definition

$$V = U \oplus W \text{ Direct Sum } \Leftrightarrow \begin{cases} V = \{u + w \, : \, u \in U \, , \, w \in W\} \\ U \cap W = \{\emptyset\} \end{cases}$$

Definition

Let X be a Banach space and $Y \subset X$ then the Orthogonal space is

$$Y^{\perp} = \{f \in X^* \, : \langle f, x \rangle = 0 \, , \forall x \in Y\}$$

Last tools: Direct Sum, Orthogonal Space, Fundamental Function

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Last tools: Direct Sum, Orthogonal Space, Fundamental Function

Definition

Let $t \in [0, |\Omega|]$, $E_t \subset \Omega$, $|E_t| = t$, then the Fundamental Function is defined as

$$\varphi_X(t) = \rho(\chi_{E_t}) = ||\chi_{E_t}||_X$$

* ff   torna a ff

Theorem

Let (Ω, μ) be a space of measure totally σ-finite and non atomic, let $X(\rho)$ be a r.i.BFS on (Ω, μ). The following conditions are equivalent

i) $\lim_{t \to 0} \varphi_X(t) = 0$

ii) $X_a = X_b$

iii) $(X_b)^* = X'$

Last tools: Direct Sum, Orthogonal Space, Fundamental Function

Definition

Let $t \in [0, |\Omega|]$, $E_t \subset \Omega$, $|E_t| = t$, then the Fundamental Function is defined as

$$\varphi_X(t) = \rho(\chi_{E_t}) = ||\chi_{E_t}||_X$$

* ff torna a ff

Theorem

Let (Ω, μ) be a space of measure totally σ-finite and non atomic, let $X(\rho)$ be a r.i.BFS on (Ω, μ). The following conditions are equivalent

i) $\lim_{t \to 0} \varphi_X(t) = 0$

ii) $X_a = X_b$

iii) $(X_b)^* = X'$

Motivation

$$(p1)+(p2)$$

$$\uparrow$$

X reflexive BFS $\Rightarrow$ X^* is isometrically isomorph to X'

* p2

EXP BFS non reflexive

$$(EXP(\Omega))' = L\log L(\Omega)$$

$$(EXP(\Omega))^* = L\log L(\Omega) \oplus (exp(\Omega))^{\perp}$$

$$\Downarrow$$

$$(EXP(\Omega))^* = (EXP(\Omega))' \oplus (EXP(\Omega))_b^{\perp}$$

where we have used $exp(\Omega) = \overline{L^\infty(\Omega)}^{EXP(\Omega)} = (EXP(\Omega))_b$

$$(p1) + (p2)$$

$$\uparrow$$

$$\text{X reflexive BFS} \Rightarrow X^* \text{ is isometrically isomorph to } X'$$

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where we have used $\quad exp(\Omega) = \overline{L^\infty(\Omega)}^{EXP(\Omega)} = (EXP(\Omega))_b$

Question

$$X \ \text{BFS} + ?$$

$$\Downarrow$$

$$X^* = X' \oplus (X_b)^{\perp}$$

Decomposition formula

Theorem

Let X be a r.i. BFS and let $\varphi_X(t)$ be its fundamental function. If

i) $\lim_{t\to 0} \varphi_X(t) = 0$

Then the following decomposition formula holds

$$X^* = X' \oplus (X_b)^{\perp}$$

* ff
* df

Alternative formulations

* e   torna a e

By the following equivalences we get

$$X_a = X_b \quad \Leftrightarrow \quad \lim_{t\to 0} \varphi_X(t) = 0 \quad \Leftrightarrow \quad (X_b)^* = X'$$

$$\Downarrow$$

$$X^* = X' \oplus (X_b)^\perp$$

$$\Updownarrow$$

$$X^* = (X_b)^* \oplus (X_b)^\perp$$

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Let X be an Orlicz space then

$$\varphi_X(t) = \frac{1}{\Phi^{-1}(1/t)} \Rightarrow \lim_{t\to 0} \varphi_X(t) = 0 \Rightarrow \text{Decomposition formula}$$

$\uparrow$

([C-Fiorenza])

$$X = EXP_\alpha(\Omega) \quad \Rightarrow \quad \begin{cases} (EXP_\alpha(\Omega))' = L\log^{\frac{1}{\alpha}} L(\Omega) \ \ ([BS]) \\ (EXP_\alpha(\Omega))_b^\perp = (exp_\alpha(\Omega))^\perp \end{cases}$$

$$\Downarrow$$

$$\begin{aligned}(EXP_\alpha(\Omega))^* &= (EXP_\alpha(\Omega))' \oplus (EXP_\alpha(\Omega))_b^\perp \\ &= L\log^{\frac{1}{\alpha}} L \oplus (exp_\alpha(\Omega))^\perp \\ &= (EXP_\alpha(\Omega))_b{}^* \oplus (EXP_\alpha(\Omega))_b{}^\perp \\ &= (exp_\alpha(\Omega))^* \oplus (exp_\alpha(\Omega))^\perp\end{aligned}$$

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$X' = (X_b)^*$ *

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$\uparrow$

$X' = (X_b)^*$ * e

Let $X = L^{p),\theta}(\Omega)$ then

$$\varphi_X(t) = t^{1/p}\left[\log\left(\frac{1}{t}\right)\right]^{-\frac{\theta}{p}} \Rightarrow \lim_{t\to 0}\varphi_X(t) = 0 \Rightarrow \text{Decomposition formula}$$

$\uparrow$

([C-Fiorenza])

$$(L^{p),\theta}(\Omega))^* = (L^{p),\theta}(\Omega))' \oplus (L_b^{p),\theta}(\Omega))^{\perp}$$

$$= L^{(p',\theta}(\Omega) \oplus (L_b^{p),\theta}(\Omega))^{\perp}$$

$\uparrow$

([C-Fiorenza])

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$\uparrow$

$$X' = (X_b)^*$$

Theorem

Let X be a BFS, then

$$X \subseteq (X')^* \tag{3}$$

The equality holds if and only if X' has absolutely continuous norm.

Proof

Let us start by proving the inclusion (3), we have

$$X\,\mathrm{BFS} \Rightarrow X'\,\mathrm{BFS} \Rightarrow (X')' \subseteq (X')^* \Rightarrow X \subseteq (X')^*$$

Let us characterize the equality in (3), to this aim let us assume X' with a.c.n. then

$$(X') = (X'_a) \Rightarrow (X')^* = (X'_a)^* = (X'_a)' = (X')' = X'' = X \Rightarrow (X')^* = X$$

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Counterexample

$$X = L^1$$

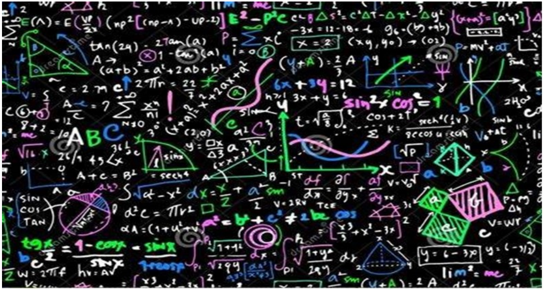
$$\Downarrow$$

$$(X')^* = ((L^1)')^* = (L^\infty)^* \supset L^1 = X$$

$$\Downarrow$$

$$L^1 = X \subset (X')^* = (L^\infty)^*$$

Let us notice that $X' = (L^1)' = L^\infty$ has not a.c.n.



Cóż, mogę na tym poprzestać, mam nadzieję, że udało mi się przedstawić ten przegląd tych różnych problemów matematycznych, prawdopodobnie zbyt wielu nowych pojęć w zbyt krótkim czasie, ale niezależnie od dostarczonych pojęć,

chodziło o to, aby opisać, jak może powstać problem matematyczny i w jaki sposób możemy wykorzystać naszą wiedzę, teoretyczne narzędzia naszej wiedzy, aby je rozwiązać, dlatego mam nadzieję, że dałem Ci trochę inspiracji, aby docenić środowisko badań matematycznych, a zatem wagę wszystkich teoretycznych koncepcji, które dodajesz podczas wykonywania swoich obowiązkowych zadań .


BARDZO DZIĘKUJĘ ZA UWAGĘ.
POWODZENIA W TWOJEJ PRZYSZŁOŚCI
I W TWOICH PRZYSZŁYCH WYBORACH!
THANK YOU VERY MUCH FOR YOUR ATTENTION
GOOD LUCK FOR YOUR FUTURE
AND FOR YOUR FUTURE CHOISES!

Example of sequence where limit and integral cannot commute

* li torna a li

$$f_n = \frac{1}{n}\chi_{(0,n)} = \begin{cases} \frac{1}{n} & (0,n) \\ 0 & \text{otherwise} \end{cases}$$

then

$$f_n(x) \to 0 \quad \text{unif.} \qquad \text{but} \qquad f_n \not\to 0 \quad \text{in } L^1$$

that is

$$\int_\Omega \lim_{n\to\infty} f_n(x)\,dx \neq \lim_{n\to\infty}\int_\Omega f_n(x)$$

Indeed

$$\int_\Omega \lim_{n\to\infty} f_n(x)\,dx = 0 \neq 1 = \lim_{n\to\infty}\int_\Omega f_n(x)$$

Proof of the decomposition formula

* df   torna a df

Let $l \in X^*$, let us set

$$\nu(F) = l\,(\chi_F)\ ,\ \forall\, F \subset \Omega \text{ measurable}$$

- σ-additive
- absolutely continuous respect to the Lebesgue measure $|F|$

$$\Downarrow$$

- ν has Radon Nikodym derivative, locally integrable

$$l(f) = \int_\Omega f\, g\, dx\ ,\ \ \forall\, f \in L^\infty$$

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Proof of the decomposition formula

* df torna a df

Let $I \in X^*$, let us set

$$\nu(F) = I\,(\chi_F)\ ,\ \forall\, F \subset \Omega \text{ measurable}$$

- σ-additive
- absolutely continuous respect to the Lebesgue measure $|F|$

$$\Downarrow$$

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$$I(f) = \int_\Omega f\, g\, dx\ ,\ \ \forall\, f \in L^\infty$$

Proof of the decomposition formula

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$$I(f) = \int_\Omega f\, g\, dx\ ,\ \ \forall\, f \in L^\infty$$

$l \in X^* \quad \Rightarrow \quad l(f) \leq K||f||_X \ , \ \forall f \in X$

$$\Downarrow$$

$$\int_\Omega f\, g\, dx \leq K\, ||f||_X \ , \ \forall f \in X$$

$$\Downarrow$$

$$g \in X'$$

For every $g \in X'$, $\quad l_g \ : f \in X_b \to \int_\Omega f\, g\, dx$

$\Downarrow$ Hölder inequality

$l_g \in (X_b)^* = X' \Rightarrow l_g \in X'$

Let l_s be defined by $l_s = l - l_g$, then

$l_s(f) = \langle l_s, f \rangle = 0 \ , \ \forall f \in X_b \Rightarrow l_s \in (X_b)^\perp$

Hence

$$l = l_g + l_s \in X' + (X_b)^\perp \ , \ X' \cap (X_b)^\perp = \{\emptyset\}$$

$l \in X^* \quad \Rightarrow \quad l(f) \le K \|f\|_X \ , \ \forall f \in X$

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Appendix Definitions

$$B \subset X^* \quad norm - fundamental$$

$$\Updownarrow$$

$$||f||_X = sup\,\{|L(f)| : L \in B\,,\ ||L||_{X^*} \leq 1\}$$

* d1 torna a d1

Order ideal

$$X \text{ BFS}, \; Y \subset X \text{ linear closed subspace, order ideal}$$

$$\Updownarrow$$

$$f \in Y, \; |g| \leq |f| \text{ a.e.} \; \Rightarrow \; g \in Y$$

* d2 torna a d2

Appendix C.Formica-Giova

([C.-Formica-Giova]) * cfg

torna a cfg

Let $1 < p < \infty$

$$L^{p),\delta}(\Omega) = \left\{ f \in L^1(\Omega) : ||f||_{p),\delta} < \infty \right\}$$

where

$$||f||_{p),\delta} = \sup_{0<\epsilon<p-1} \left(\delta(\epsilon)^{\frac{1}{p-\epsilon}} \int_\Omega |f|^{p-\epsilon} \right)^{\frac{1}{p-\epsilon}}$$

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$$B_p := \begin{cases} \delta(0+) = 0 \\ 0 < \delta \leq 1 \\ \delta(\epsilon)^{\frac{1}{p-\epsilon}} \text{ increasing in } \epsilon \end{cases}$$

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